Use of Two-Dimensional Representations to aid the Transition from Arithmetic to Algebra

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> Aim

To investigate the extent to which use of a teaching approach based on the concept of area would help students overcome key difficulties in the transition from arithmetic to algebra.

> Dimensions of this Case Study

Nine classes of Year 7 students across a split-site College were tested before and after the six week period of instruction: five classes formed the experimental group on one site and the remaining four classes, situated on another site provided the control group.

> Summary of Findings for this Case Study

- A comparison between pre- and post-tuition tests showed a mean improvement of 8.5% for the control group and 33.6% for the experimental group.
- The experimental group showed greater gains in both the boys' and the girls' performance than the control group but there was a particularly enhanced effect on boys in lower sets.
- The results indicated that formal numerical understanding was increasingly necessary for formal algebraic understanding as questions became more structurally complex and the algebraic operations to be performed became more demanding.
- The connection between arithmetic and algebra gives some indication that in order to enhance 'value added' in later years, the investment in developing the students' understanding of the underlying principles, emphasised by the National Curriculum and the National Numeracy Strategy, was essential.
- Some of the principles which were found to be crucial to success in algebra, such as the correct use of the equals sign and the maintaining of equivalence, have links with other aspects of mathematics, suggesting that the approach used here could have more widespread impact.

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Introduction

The revised National Curriculum at all Key Stages emphasises the importance of making connections between different aspects of mathematics. It advocates teaching of algebra that is firmly rooted in a solid understanding of arithmetic. The National Curriculum emphasises the structure of arithmetic by stating that 'pupils should be taught to':

- '...use the symbol = to represent equality' [Key Stage 1];
- '...understand why the commutative, associative and distributive laws apply to addition and multiplication and how they can be used to do mental and written calculations more efficiently' [Key Stage 2]; and
- 'use brackets and the hierarchy of operations' [Key Stage 3].

The National Curriculum, at Key Stage 3, then emphasises the transition from arithmetic to algebra by stating that 'pupils should be taught to':

- 'distinguish the different roles played by letter symbols in algebra';
- 'understand that the transformation of algebraic expressions obeys and generalises the rules of arithmetic'; and
- 'simplify or transform algebraic expressions'.

The National Numeracy Strategy framework for Year 7 echoes a similar theme stating that pupils should:

- 'Use the equals sign appropriately and correctly' [learning outcome: algebra];
- 'Consolidate understanding of the operations of multiplication and division, their relationship to each other and to addition and subtraction and of the principles (not the names) of the arithmetic laws' [teaching objective: calculations];
- 'Reinforce the idea of an unknown' [learning outcome: algebra]; and
- 'Know that algebraic operations follow the same conventions and order as arithmetic operations' [key teaching objective: algebra].

Research shows that many children are hampered in their study of algebra by:

- a limited appreciation of the meaning of the equals sign;
- a lack of understanding of the structure of arithmetic;
- a limited understanding of an unknown;
- inability to perform operations using unknowns; and
- a focus on surface structure, such as syntax, at the expense of underlying meaning.

These hurdles to understanding are addressed and emphasised by both the revised National Curriculum and the National Numeracy Strategy.

Both the National Curriculum and the complementary National Numeracy Strategy objectives outlined above are reflected in the methodology of the experimental approach that is adopted in this research. Here we attempted to:

- reinforce understanding of equality and use of the equals sign;
- reinforce the structure of arithmetic;
- facilitate progression in understanding of an unknown;
- enable students to perform algebraic operations;
- enable students to understand an expression based on its underlying arithmetical structure; and
- prevent students' workings becoming detached from the teaching model.

Approach to the Teaching of Algebra using the concept of area

A key objective for Year 6 from the National Numeracy Strategy is that pupils should be able to 'calculate the area of simple compound shapes that can be split into rectangles.' We developed the teaching model to meet this objective. It was intended to illustrate the structure of an expression or equation by representing it visually in two-dimensional form, thus preventing students from concentrating solely on its syntax.

The model was first introduced in the context of arithmetic equalities (such as $1 + 2 \times 3 = 7$), leading on to equations formed from those equalities (such as $1 + 2x = 7$), where the unknown has an identifiable role, and subsequently on to a variety of linear equations (Fig. 1). Once an unknown was introduced, students were encouraged to draw diagrams freehand, not to scale. Equations with an unknown on both sides, requiring the students to understand and operate on the unknown, were only introduced after the students became well-practised at equations with the unknown on one side only.

The model was then used to introduce more complex linear equations, substitution and simplification of expressions, involving the four operations and the use of brackets.

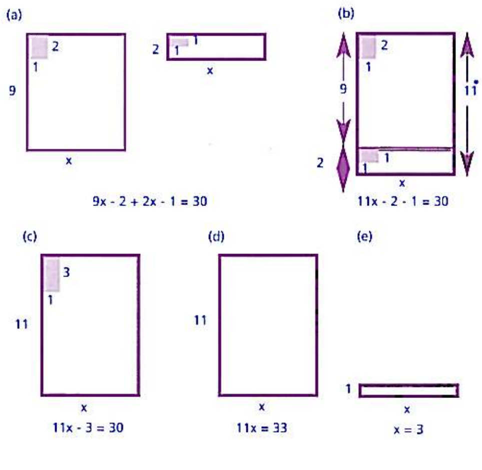

Figure 1: *Use of the model to solve the equation $9x - 2 + 2x - 1 = 30$*

This approach involved placing particular emphasis on linking each diagram to its arithmetic or algebraic form to avoid detachment from the model. The model provided a constant illustration of the algebraic transformations employed in solving equations without obscuring the sequence of equations involved in the solution process.

Once students gained sufficient confidence, they dispensed with the diagrams and wrote only the sequence of equations. When explaining their work to others, students could describe the diagram that would have been drawn at any stage without actually having the need to draw it. This was intended to reinforce the use of the equals sign by encouraging students to write a separate equation for each line of working.

Details of the Study

The study was conducted with Year 7 pupils on two sites of our split-site College. Students on one site formed the experimental group and those on the other site formed the control group. Students had been placed into ability groups, within each site, immediately prior to the teaching experiment.

A 'teacher guide' describing the background to and details of the new teaching method was produced for the teachers of the experimental groups. Each teacher attended two meetings: one at the beginning of the experiment and one part way through. The sessions enabled them to work with the method and to discuss practicalities of its use for different ability groups. The Mathematics department at King Alfred's attempts to maintain the balance of sets taught by each teacher. Sets had been assigned prior to the commencement of this study, according to this policy, ensuring as near a random allocation as possible, for the purposes of this research. Following discussion, it was decided to exclude the lowest ability sets from both sites, leaving five experimental sets and four control sets. These nine classes, comprising 249 students and nine teachers took part in this experiment.

Students were assessed on arithmetic, equations and expressions both before and after the teaching intervention. The tests were designed to highlight the misconceptions detailed in the literature as particular hurdles to understanding.

For example:

One of these equations has the same value for x as 15x - 2 = 28.

Which equation is it? Circle the appropriate letter:

A) 15x = 26 B) 15x - 12 = 38 C) 13x = 28

D) 15x - 6 = 24 E) 15x - 1 = 27

Option A, for example, tests for ignoring the position of the equals sign, while the error of adding x-terms to constant terms is tested by option C. This has been referred to as the 'usual error' by Filloy and Rojano (1985) and is considered by them a major problem when models are used to support the learning process. Questions were not biased towards a particular teaching method and did not include any diagrams or reference to area. Most questions required the student to complete a given line of working out or to simplify expressions. Where questions were slightly non-standard, such as the example above, specific instructions were given to all teachers involved not to expose students to their style beforehand.

Results

All test responses were entered into an Excel spreadsheet for 'marking' and analysis. Statistical software (SPSS) was also used to determine the comparability of the test and control groups before tuition, the overall effectiveness of the approach and to highlight the influential variables.

- In a given test more students repeated the same type of error consistently in all relevant questions, than would be expected by chance, indicating a misconception. However in similar tests following tuition the same type of error was not seen, indicating that the students no longer held these misconceptions. In some cases old misconceptions had been replaced with new ones.
- The mean score for the control group rose from 20.9% (pre-test) to 28.5% (post-test). In comparison, the mean score obtained by the experimental group increased from 16.6% to 50.8%. This pattern was mirrored in the median scores.

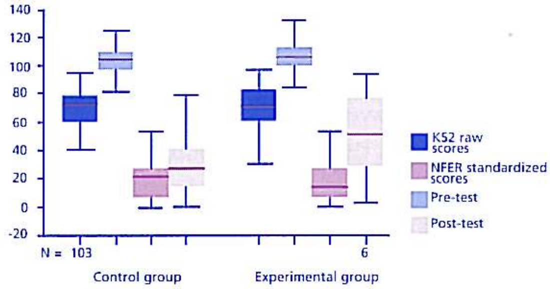

Figure 2: *Control and Experimental groups before and after teaching intervention*

The box and whisker plot of Figure 2 shows the comparability of the groups before the study, alongside the post-test results. Before the intervention, the groups were well matched on upper and lower quartiles, although the experimental group had lower median scores for Key Stage 2 and the pre-test. Following the study, the post-test scores were substantially higher in the experimental group than in the control group.

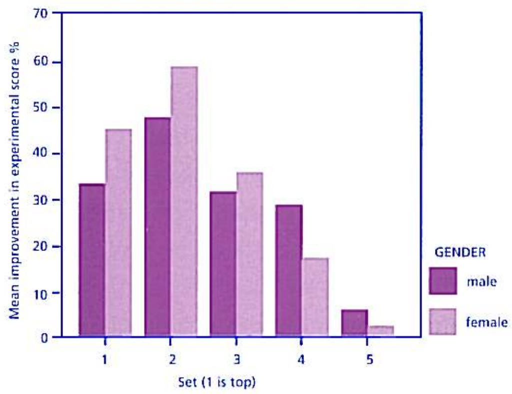

Figure3: *Improvement in the experimental group scores by set and gender*

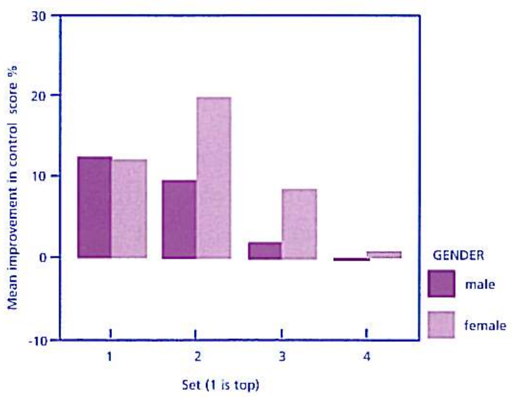

Figure 4: *Improvement in the control group scores by set and gender*

One of the most interesting findings of this research was the comparative improvement of girls and boys. Figures 3 and 4 show the improvement between pre- and post-tests by set and by gender. In the control group, boys improved slightly more than girls in the top set but there was a progressive drop in the boys' improvement for sets 2, 3 and 4. The ratio of the boys' improvements to girls' improvements decreased with set number (set 1 is top). This situation was completely reversed in the experimental group with the boys making a relative gain over girls. The ratio of boys' improvements to girls' improvements increases with set number. This may have been due to the boys possibly accessing the greater spatial awareness potential often evident among boys.

The experimental approach was extremely successful at overcoming the documented hurdles to understanding, even those considered problematic for representational methods, such as adding x terms to constant terms. The approach was successful, during the period of study, at introducing arithmetical structure and a progressively more general understanding of the unknown. A spot-check of one of the experimental sets, five months after the conclusion of the study, showed that they had retained the ability to operate on an unknown in an unfamiliar context. Although encouraging, a systematic follow-up study would be required to assess the long-term effects of the experimental approach.

An ability both to use correctly and understand the equals sign and arithmetic operations to maintain equivalence was linked to the ability to identify the next step in solving an equation and to simplifying expressions.

An increasing link between such arithmetical structure and success at progressively-complex equations and expressions in both control and experimental groups was found: whilst understanding no deeper than surface structure was rewarded by success on relatively easy questions whose surface structure was familiar, a block to further progress was evident when questions were more complex. These findings are especially important if students are to progress fluently to questions requiring them to solve an equation they have established for a specific purpose, such as determining the radius of a cylinder given its volume.

Further investigation is needed to determine the extent to which these skills might be of benefit to other topics, where the same issues of producing a rigorous, next stage of working, rewriting an expression or equation in an equivalent form and using the equals sign correctly, are of central importance.

Towards a Structural Understanding of Mathematics

Although the original National Curriculum was often interpreted in a fragmentary fashion, this has been superseded by an ethos of structure in its recent revision. This is reflected in the rigorous approach to the understanding of underlying principles and the linking of these principles to the different aspects of mathematics. The method trialled in this research was designed to complement such a structural approach and the principles it engendered may be widely applicable throughout the mathematics curriculum.

Further Reading

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